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UNIFORM HEATING OF A LOCALLY INHOMOGENEOUS ELASTIC PLATE†

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The plane problem of the uniform heating of a locally inhomogeneous elastic isotropic plate is solved. The inhomogeneity of the material in question is concentrated in a circle or a ring, and the thermomechanical characteristics of the material have circular symmetry and vary sharply inside this circle or ring. The functions representing the properties of the material are continuous everywhere (together with its first derivatives) and take constant values outside the region of local inhomogeneity. The case of an infinite plate is considered. In addition, the temperature problem for a finite plate, when the dimensions of this plate are much greater than the dimensions of the region of inhomogeneity, is also solved. A semianalytic method of solving such temperature problems is proposed. © 1996 Elsevier Science Ltd. All rights reserved.

The majority of papers published on the theory of elasticity for inhomogeneous solids are devoted to materials with a uniform inclusion, and also piecewise-uniform materials. Structures having a locally continuous inhomogeneity of the mechanical properties have been somewhat less investigated, particularly in cases when the mechanical properties of the body are described by rapidly varying functions [1–3].

In this paper we consider the elastic equilibrium of a locally inhomogeneous isotropic plate when the temperature is uniformly distributed. The inhomogeneity of the material is concentrated in a circle of radius a, and the thermomechanical characteristics of the material have circular symmetry and vary rapidly inside this circle.

The origin of coordinates is places at the centre of the circle. We will solve the axisymmetric problem in a polar system of coordinates r, θ. We will use the dimensionless polar radius r (relative to a).

We will write the relation between the stresses σ_r, σ_θ and the strains ε_r, ε_θ for the case of a plane stress state as follows:

$$E(r)\begin{Bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{Bmatrix}=\begin{Bmatrix}\sigma_r\\ \sigma_\theta\end{Bmatrix}-\nu(r)\begin{Bmatrix}\sigma_\theta\\ \sigma_r\end{Bmatrix}+E(r)\,\alpha(r)\,\Delta T$$

$$G(r)=\frac{E(r)}{2(1+\nu(r))},\quad \Delta T=\text{const} \tag{1}$$

where $E(r)$, $G(r)$, $\nu(r)$ and $\alpha(r)$ are the variable Young's modulus, shear modulus, Poisson's ratio and the coefficient of thermal expansion, respectively.

We will specify the thermomechanical characteristics of the material as follows:

$$\xi(r)=\xi_1+(\xi_0-\xi_1)\,f(r-r_0),\quad \xi=E,G,\alpha \tag{2}$$

The subscript one denotes the elastic constants in the region $r\geqslant 1$ and the zero subscript denotes the elastic constants in the region $r\leqslant r_0$, $1>r_0>0$. The function $f(r-r_0)=0$ when $r\geqslant 1$, $f(r-r_0)=0$ when $r\leqslant r_0$, while in the region $r_0\leqslant r\leqslant 1$

$$f(y)=\{s_1\,\text{ch}(\beta y)\cos(\beta y)+s_2\,\text{sh}(\beta y)\sin(\beta y)+1\}/(s_1+1),\quad y=r-r_0$$

The constants s_1 and s_2 are given; they are subject to the requirement $f'(1-r_0)=f(1-r_0)=0$. The conditions $f'(0)=0$ and $f(0)=1$ are then automatically satisfied. The function f and its first derivative are continuous everywhere. The parameter β satisfies the condition $\beta^{-1}\ll 1$. In this case, the function

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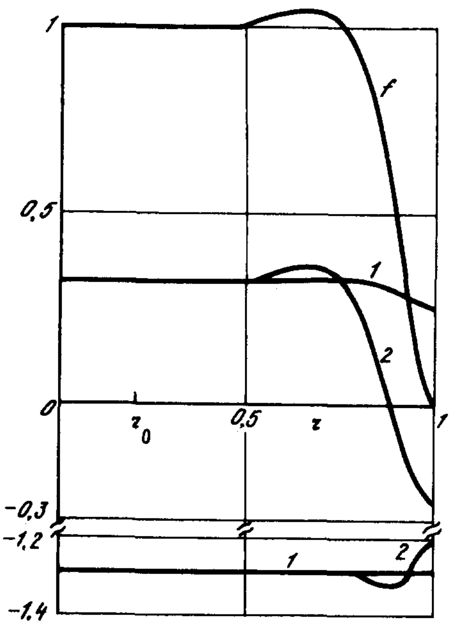

Fig. 1.

f varies rapidly in the region of the boundary $r = 1$ (Fig. 1, curve f). Such a distribution of the thermomechanical characteristics (when $E_0 > E_1$) may, for example, be the result of cold hardening of a circular plate $r \leqslant a_1$ from porous material with a piecewise-constant thickness. The ring $a_0 \leqslant r \leqslant a_1$ has a thickness h_1, and the region $r \leqslant a_0$ has, correspondingly, a thickness $h_0 > h_1$. The purpose of cold hardening is to obtain a plate of uniform thickness $h_* < h_1$.

We will write the equations of equilibrium and of compatibility of the strains for the case of the axisymmetric problem in the form

$$\sigma_\theta = r\frac{d\sigma_r}{dr} + \sigma_r, \quad r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r = 0 \tag{3}$$

In an infinite plate the strains ε_θ, ε_r approach zero as $r \to \infty$, while the functions σ_θ and σ_r are finite at the point $r = 0$.

The mechanical characteristics of the material are constant outside the region $1 \geqslant r \geqslant r_0$. Consequently, in the region $0 \leqslant r \leqslant r_0$ the stresses are constant: $\sigma_r = \sigma_\theta - \sigma_r(r_0)$, while in the region $r \geqslant 1$ the stress distribution has the form

$$\sigma_r = [\sigma_r(1) + P(1-\nu_1)^{-1}]\, r^{-2} - P(1-\nu_1)^{-1}, \quad P = E_1\alpha_1\Delta T$$
$$\sigma_\theta = -[\sigma_r(1) + P(1-\nu_1)^{-1}]\, r^{-2} - P(1-\nu_1)^{-1} \tag{4}$$

For an infinite plate, instead of stress distribution (4) we will have, in the region $r \geqslant 1$, the solution

$$\sigma_r = [\sigma_r(1) + A]\, r^{-2} - A, \quad A = \sigma_r(1)/(r_*^2 - 1)$$
$$\sigma_\theta = -[\sigma_r(1) + A]\, r^{-2} - A \tag{5}$$

where $r = r_*$ is the free boundary of the plate.

Hence, the problem reduces to solving system (1)–(3) in the region $1 \geqslant r \geqslant r_0$ with the following boundary conditions

$$r = 1: \sigma = -2P(1-\nu_1)^{-1}; \quad r = r_0: \sigma = 2\sigma_r \tag{6}$$

or

$$r = 1\colon \sigma = -2\sigma_r/(r_*^2 - 1); \quad r = r_0\colon \sigma = 2\sigma_r \tag{7}$$

where $\sigma = \sigma_r + \sigma_\theta$.

Condition (7) corresponds to the solution of the temperature problem for a finite plate ($r = r_*$ is the free boundary).

Note that when $r_*^{-1} \ll 1$, we have $\sigma(1) \approx 0$, and the stress state in the region $r \leqslant 1$ is practically independent of the location of the boundary (i.e. is independent of r_*).

Relations (6) and (7) follow from the condition for the stresses and strains to be continuous on the boundaries $r = 1$ and $r = r_0$, taking into account the nature of the stress state outside the region $1 \leqslant r \leqslant r_0$.

We will describe the method of solving problems (1)–(3). We will split the region $[r_0, 1]$ into N equal sections $[\eta_n, \eta_{n+1}]$, $n = 0, \ldots, N-1$; $\eta_0 = r_0$, $\eta_N = 1$.

We will write the system of equations (3) in the following integral form

$$\begin{aligned} C(\eta) &= C(\eta_n)(\eta/\eta_n)^{-2} + \tfrac{1}{2}\eta^{-2} L_1[C(\eta)]_{\eta_n}^{\eta} + \\ &+ D(\eta_n)\,\eta^{-2} \int_{\eta_n}^{\eta} r \frac{E(r)}{E(\eta_n)}\,dr - \eta^{-2} \int_{\eta_n}^{\eta} r E(r)\,[\alpha(r) - \alpha(\eta_n)]\,\Delta T\,dr \\ D(\eta) &= D(\eta_n)\,E(\eta)/E(\eta_n) - E(\eta)\,[\alpha(\eta) - \alpha(\eta_n)]\,\Delta T + \tfrac{1}{2} E(\eta)\, L_2[C(\eta)]_{\eta_n}^{\eta} \end{aligned} \tag{8}$$

Here

$$C(\eta) = \sigma_r(\eta), \quad D(\eta) = \sigma_r(\eta) + \sigma_\theta(\eta), \quad r \in [\eta_n, \eta_{n+1}]$$

$$L_1[C(\eta)]_{\eta_n}^{\eta} = \int_{\eta_n}^{\eta} E(r)\, r \int_{\eta_n}^{r} \left(\frac{1}{G(\xi)}\right)' C(\xi)\,d\xi\,dr$$

$$L_2[C(\eta)]_{\eta_n}^{\eta} = \int_{\eta_n}^{\eta} \left(\frac{1}{G(\xi)}\right)' C(\xi)\,d\xi$$

We will obtain the general solution of the initial homogeneous system of equations (3) and also its partial solution.

We first calculate the following functions: $C_n^{(1)}(\eta)$—the solution of the homogeneous system of equations (8) in the section $[\eta_n, \eta_{n+1}]$ with boundary conditions $C_n^{(1)}(\eta_{n+1}) = 1$, $D_n^{(1)}(\eta_{n+1}) = 0$; $C^{(2)}{}_n(\eta)$—the solution of system (8) with the boundary conditions

$$C_n^{(2)}(\eta_{n+1}) = 0, \quad D_n^{(2)}(\eta_{n+1}) = 1;$$

$C_n^{(0)}(\eta)$—the partial solution of system (8) in the section $[\eta_n, \eta_{n+1}]$ for the conditions $C_n^{(0)}(\eta_{n+1}) = 0$, $D_n^{(0)}(\eta_{n+1}) = 0$.

These functions can be found by the method of successive approximations, solving system (8) in the small section $[\eta_n, \eta_{n+1}]$. When integrating system (8) it is assumed that the integral terms L_1 and L_2 are known; they are found in the previous step of the iterative process. When finding the first approximation, the terms L_1 and L_2 are assumed to be zero.

We will obtain the solution of the homogeneous system of equations (8).

We substitute the solution $C(\eta) = C(\eta_{n+1})\, C_n^{(1)}(\eta) + D(\eta_{n+1})\, C_n^{(2)}\eta$ into the integral terms L_2 and L_2 of system (8) in the section $[\eta_n, \eta_{n+1}]$, and rewrite the homogeneous system of equations (8) with $\eta = \eta_{n+1}$ in the form

$$\begin{aligned} C(\eta_{n+1}) &= C(\eta_n)(\eta_{n+1}/\eta_n)^{-2} + \tfrac{1}{2}\eta_{n+1}^{-2}[C(\eta_{n+1})\,\varphi_{1,n} + \\ &+ D(\eta_{n+1})\,\varphi_{2,n}] + D(\eta_n)\,\eta_{n+1}^{-2} \int_{\eta_n}^{\eta_{n+1}} r\,\frac{E(r)}{E(\eta_n)}\,dr \end{aligned} \tag{9}$$

$$D(\eta_{n+1}) = D(\eta_n)\, E(\eta_{n+1}) / E(\eta_n) + \tfrac{1}{2} E(\eta_{n+1}) [C(\eta_{n+1})\, \psi_{1,n} + D(\eta_{n+1})\, \psi_{2,n}]$$

$$\varphi_{i,n} = L_1 [C_n^{(i)}(\eta)]_{\eta_n}^{\eta_{n+1}}, \quad \psi_{i,n} = L_2 [C_n^{(i)}(\eta)]_{\eta_n}^{\eta_{n+1}}, \quad i = 0, 1, 2$$

System (9) enables us to determine the required quantities $C(\eta_{n+1})$, and $D(\eta_{n+1})$ for known values of $C(\eta_n)$, and $D(\eta)$. Calculations were carried out using the implicit scheme [4], which guarantees that the calculation will be stable. The solution of the system of linear algebraic equations (9) is the general solution of the initial homogeneous system (3) at the nodes $n = 0, 1, 2, \ldots, N$, i.e. this solution depends linearly on the initial conditions $C(r_0)$ and $D(r_0)$.

To obtain the partial solution $C_*(\eta_{n+1})$, and $D_*(\eta_{n+1})$ of system (3) we must add the terms

$$\frac{1}{2}\eta_{n+1}^{-2}\varphi_{0,n} - \eta_{n+1}^{-2} \int\limits_{\eta_n}^{\eta_{n+1}} r E(r) [\alpha(r) - \alpha(\eta_n)]\, \Delta T\, dr$$

and

$$\tfrac{1}{2} E(\eta_{n+1})\, \psi_{0,n} - E(\eta_{n+1}) [\alpha(\eta_{n+1}) - \alpha(\eta_n)]\, \Delta T$$

to the right-hand sides of the equations of system (9), respectively, and redenote the quantities $C(\cdot)$ and $D(\cdot)$ by $C_*(\cdot)$ and $D_*(\cdot)$.

Note that in this case the operators L_1 and L_2 are applied to the solution

$$C_*(\eta) = C_*(\eta_{n+1})\, C_n^{(1)}(\eta) + D_*(\eta_{n+1})\, C_n^{(2)}(\eta) + C_n^{(0)}(\eta)$$

We assume that the partial solution satisfies the condition

$$C_*(r_0) = D_*(r_0) = 0$$

We write the general solution of the problem in the form

$$\sigma_r = BC(\eta) + C_*(\eta), \quad \sigma_\theta = B[D(\eta) - C(\eta)] + D_*(\eta) - C_*(\eta)$$

where the solution $C(r)$ and $D(r)$ of the homogeneous system of equations is specified by the condition $C(r_0) = 1$ and $D(r_0) = 2$. In this case the second boundary condition (6) or (7) is satisfied directly. The constant B is found from the first boundary condition (6) or (7).

Hence, we have formally constructed an exact solution of system (3) with conditions (6) or (7).

This algorithm was realized on a personal computer. To apply the method of successive approximations we use a subroutine for the numerical integration of functions. Calculations show that the semianalytic method of solving the temperature problem proposed here is highly accurate even for a small number M of iterations (for example, for $M = 3$ or $M = 2$) using the method of successive approximations to calculate the functions $C^{(i)}{}_n(\eta)$, $i = 0, 1, 2$ in the small interval $[\eta_n, \eta_{n+1}]$.

In Fig. 1 the upper curves represent the stress distribution in the finite plate ($r_* = 100$): $\sigma_r^* = \sigma_r/P$ (curve 1), $\sigma_\theta^* = \sigma_\theta/P$ (curve 2) and of the functions $f(r - r_0)$ (curve f) with respect to r in the region $r \leqslant 1$ for $k = 0.5$, $c_\alpha = -0.05$, $\nu_1 = 0.47$, $r_0 = 0.2$, and $\beta = 10$, where

$$P = E_1 \alpha_1 \Delta T, \quad k = \frac{E_0 - E_1}{E_1}, \quad c_\alpha = \frac{\alpha_0 - \alpha_1}{\alpha_1}$$

For the same values of the parameters, the stress distribution in an infinite plate is shown by the lower curves (σ_r^* is curve 1).

In the region $r > 1$ the stress distribution is described by (4).

The temperature problem ($\Delta T = \text{const}$) was solved similarly for an inhomogeneous plate in the case when the inhomogeneity of the material is localized inside the ring $r_0 \leqslant r \leqslant 1$. The case of a finite circular plate $r_*^{-1} \ll 1$ is considered.

The thermomechanical characteristics of the material are specified by relations similar to (2), namely

$$\xi(r) = \xi_1 + (\xi_2 - \xi_1)\, f(r - r_2), \quad r_2 = (1 + r_0)/2, \quad \xi = E, G, \alpha$$

where the subscript one denotes the elasticity constants in the regions $r \geqslant 1$, $r \leqslant r_0$, while the subscript

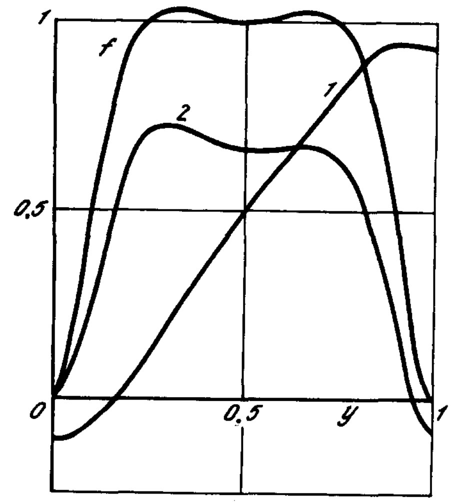

Fig. 2.

two denotes the elasticity constants in the middle surface $r = r_2$ of the ring of inhomogeneity.

The function $f(r - r_2) = 0$ for $r \geqslant 1$, $r \leqslant r_0$, but in the region $r_0 \leqslant r \leqslant 1$, as in the previous problem, it is specified by the analytic formula $f(y)$ written above, in which $y = r - r_2$.

The constants s_1 and s_2 are known; they are subject to the requirement $f'(1 - r_2) = f(1 - r_2)_0$. The conditions $f'(r_0 - r_2) = f(r_0 - r_2) = 0$ and also $f(0) = 1$ are then automatically satisfied.

We will consider the case when $\beta^{-1} \ll 1$.

In the region $0 \leqslant r \leqslant r_0$ the stresses are constant: $\sigma_r = \sigma_\theta = \sigma_r(r_0)$, while in the region $r_* \geqslant r \geqslant 1$ the stress distribution is given by (5).

In Fig. 2 we show the function $f(r - r_2)$ (curve f) and the distribution of the stresses $\sigma_r^* = 10 \times \sigma_r/P$ (curve 1) and $\sigma_\theta^* = \sigma_\theta/P$ (curve 2) with respect to two in the ring $r_0 \leqslant r \leqslant 1$ where the perturbation of the thermomechanical characteristics of the material are localized, for $k = 0.5$, $c_\alpha = -0.5$, $\nu_1 = 0.2$, $\nu_2 = 0.47$, $r_0 = 0.8$, $\beta = 10/(1 - r_0)$ and $r_* = 100$. Here $P = E_1\alpha_1\Delta T$, $k = (E_2 - E_1)/E_1$, $c_\alpha = (\alpha_2 - \alpha_1)/\alpha_1$. We have plotted the quantity $y = (r - r_0)/(1 - r_0)$ along the abscissa axis.

Note that we can obtain an exact analytic solution of the temperature problem in the case when the thermodynamic characteristics are given in the form

$$G = \text{const}, \quad E(r) = 2G(1 + \nu(r)), \quad \nu = \nu(r) \leqslant \tfrac{1}{2}, \quad \alpha = \alpha(r)$$

(see (8) with $n = 0$).

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